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Results in Mathematics

Stability of the Drygas Functional Equation on Restricted Domain

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Abstract. We study the stability of the Drygas functional equation on a restricted domain. The main tool used in the proofs is the fixed point theorem for functional spaces.

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1. Introduction

We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the *Drygas equation* iff

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \qquad x, y \in \mathbb{R}. \tag{1}$$

The above equation was introduced in [3] in order to obtain a characterization of the quasi-inner-product spaces. Ebanks, Kannappan and Sahoo in [4] have obtained the general solution of the Eq. (1) as

$$f(x) = a(x) + q(x), \qquad x \in \mathbb{R},$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is an additive function and $q\colon \mathbb{R} \to \mathbb{R}$ is a quadratic function, i.e. q satisfies the quadratic functional equation

$$q(x+y) + q(x-y) = 2q(x) + 2q(y), \qquad x, y \in \mathbb{R}.$$

A set-valued version of Eq. (1) was considered by Smajdor in [8].

The stability in the Hyers–Ulam sense of the Drygas equation has been investigated by Jung and Sahoo in [6]. They have proved that if a function $f\colon X \to Y$, where X is a real vector space and Y is a Banach space satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \le \varepsilon, \qquad x, y \in X \tag{2}$$

for same $\varepsilon > 0$, then there exists a unique additive function $a\colon X \to Y$ and and a unique quadratic function $q\colon X \to Y$ such that

$$\|f(x) - a(x) - q(x)\| \le \frac{25}{3}\varepsilon, \qquad x \in X.$$

Their result was improved first by Yang in [9] and later by Sikorska in [7]. In the case when X is an Abelian group they obtained sharper bounds: $\frac{3}{2}\varepsilon$ and ε respectively instead of $\frac{25}{3}\varepsilon$ (cf. Proposition 1 in [9] and Theorem 3.2 in [7]). The stability and solution of the Drygas equation under some additional conditions was also studied by Forti and Sikorska in [5] in the case when X and Y are amenable groups.

In the paper we present the stability results for the Drygas equation on restricted domain. Let X be a nonempty subset of a normed space and Y be a normed space. We say that a function $f\colon X \to Y$ satisfies the Drygas functional equation on X if

$$f(x+y)+f(x-y) = 2f(x)+f(y)+f(-y), \quad x, y \in X,\ x+y,\ x-y \in X. \tag{3}$$

One of the method of the proof is based on a fixed point result that can be derived from [1] (Theorem 1). To present it we need the following three hypothesis:

(H1) X is a nonempty set, Y is a Banach space, $f_1, \dots, f_k\colon X \to X$ and $L_1, \dots, L_k\colon X \to \mathbb{R}_+$ are given.

(H2) $\mathcal{T}\colon Y^X \to Y^X$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \le \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|$$

for all $\xi, \mu \in Y^X,\ x \in X$.

(H3) $\Lambda\colon {\mathbb{R}_+}^X \to {\mathbb{R}_+}^X$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \qquad \delta \in {\mathbb{R}_+}^X, x \in X.$$

Now we are in a position to present the above mentioned fixed point theorem.

Theorem 1. *Let hypotheses* (H1)–(H3) *be valid and functions* $\varepsilon\colon X \to \mathbb{R}_+$ *and* $\varphi : X \to Y$ *fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \le \varepsilon(x), \qquad x \in X,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x) < \infty, \qquad x \in X.$$

Then there exists a unique fixed point ψ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \le \varepsilon^*(x), \qquad x \in X.$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \qquad x \in X.$$

Throughout the paper $\mathbb{N}_0$ denotes the set of all non-negative integers.

2. Stability Results

Theorem 2. *Let X be a subset with 0 of a normed space, Y be a Banach space and $c \geq 0$. Assume that $p > 0$ and a function $f: X \to Y$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq c(\|x\|^p + \|y\|^p) \tag{4}$$

for all $x, y \in X$ such that $x+y, x-y \in X$.

(1) *If $p > 2$ and $-x, \frac{x}{2} \in X$ for all $x \in X$, then there exists a function $g: X \to Y$ satisfying the Drygas equation on X such that*

$$\|f(x) - g(x)\| \leq \frac{2c}{2^p - 4}\|x\|^p, \qquad x \in X.$$

(2) *If $0 < p < 1$ and $-x, 2x \in X$ for all $x \in X$, then there exists a function $g: X \to Y$ satisfying the Drygas equation on X such that*

$$\|f(x) - g(x)\| \leq \frac{2c}{2 - 2^p}\|x\|^p, \qquad x \in X.$$

(3) *If $1 < p < 2$ and $-x, \frac{1}{2}x, 2x \in X$ for all $x \in X$, then there exists a function $g: X \to Y$ satisfying the Drygas equation on X such that*

$$\|f(x) - g(x)\| \leq \left(\frac{2c}{4 - 2^p} + \frac{2c}{2^p - 2}\right)\|x\|^p, \qquad x \in X.$$

Moreover, g is the unique solution of the Eq. (3) such that $\|f(x) - g(x)\| \leq M\|x\|^p$ for all $x \in X$ and some $M \geq 0$.

Proof. First observe that the inequality (4) clearly forces $f(0) = 0$.

(1) Replacing x and y by $\frac{x}{2}$ in (4) we obtain

$$\left\| f(x) - 3f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right) \right\| \leq \frac{2c}{2^p}\|x\|^p, \qquad x \in X. \tag{5}$$

Consider functions $\mathcal{T}: Y^X \to Y^X$ and $\varepsilon: X \to \mathbb{R}_+$ given as follows

$$\mathcal{T}\xi(x) = 3\xi\left(\frac{x}{2}\right) + \xi\left(-\frac{x}{2}\right), \qquad x \in X,\ \xi \in Y^X$$

and

$$\varepsilon(x) = \frac{2c}{2^p}\|x\|^p, \qquad x \in X.$$

The inequality (5) now becomes

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon(x), \qquad x \in X.$$

For every $\xi, \mu \in Y^X$ and $x \in X$

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq 3\left\|\xi\left(\frac{x}{2}\right) - \mu\left(\frac{x}{2}\right)\right\| + \left\|\xi\left(-\frac{x}{2}\right) - \mu\left(-\frac{x}{2}\right)\right\|,$$

so $\mathcal{T}$ satisfies the inequality (H2) with $f_1(x) = \frac{x}{2}, f_2(x) = -\frac{x}{2}, L_1(x) = 3$, $L_2(x) = 1,\ x \in X$. By (H3), the operator $\Lambda\colon \mathbb{R}_+^X \to \mathbb{R}_+^X$ is given by

$$\Lambda\eta(x) = 3\eta\left(\frac{x}{2}\right) + \eta\left(-\frac{x}{2}\right), \qquad x \in X,\ \eta \in \mathbb{R}_+^X.$$

In particular

$$\Lambda\varepsilon(x) = 4\varepsilon\left(\frac{x}{2}\right) = \frac{4}{2^p}\varepsilon(x), \qquad x \in X.$$

Since Λ is linear, we can prove by induction

$$\Lambda^n\varepsilon(x) = \left(\frac{4}{2^p}\right)^n \varepsilon(x), \qquad x \in X,\ n \in \mathbb{N}_0.$$

As $p > 2$ we have $\frac{4}{2^p} < 1$. Consequently the series $\sum\limits_{n=0}^{\infty} \Lambda^n\varepsilon(x)$ is convergent for every $x \in X$ and

$$\varepsilon^*(x) = \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x) = \sum_{n=0}^{\infty} \left(\frac{4}{2^p}\right)^n \varepsilon(x) = \frac{2^p}{2^p - 4}\varepsilon(x) = \frac{2c}{2^p - 4}\|x\|^p, \quad x \in X.$$

By Theorem 1, there exists a function $g\colon X \to Y$ such

$$g(x) = \lim_{n\to\infty} \mathcal{T}^n f(x), \qquad x \in X,$$

$$g(x) = 3g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right), \qquad x \in X$$

and

$$\|f(x) - g(x)\| \leq \frac{2c}{2^p - 4}\|x\|^p, \qquad x \in X.$$

Next we prove that g satisfies the Drygas equation. Observe first that if a function $h\colon X \to Y$ satisfies the inequality

$$\|h(x+y) + h(x-y) - 2h(x) - h(y) - h(-y)\| \leq M(\|x\|^p + \|y\|^p) \tag{6}$$

for all $x, y \in X$ such that $x+y, x-y \in X$ and some $M > 0$, then

$$\|\mathcal{T}h(x+y) + \mathcal{T}h(x-y) - 2\mathcal{T}h(x) - \mathcal{T}h(y) - \mathcal{T}h(-y)\| \leq \frac{4M}{2^p}(\|x\|^p + \|y\|^p),$$

for $x, y \in X$ satisfying $x+y, x-y \in X$. Indeed, fix $h\colon X \to Y$ and assume (6). Then

$$\begin{aligned}
&\mathcal{T}h(x+y) + \mathcal{T}h(x-y) - 2\mathcal{T}h(x) - \mathcal{T}h(y) - \mathcal{T}h(-y)\\
&\quad = 3\left(h\left(\frac{x+y}{2}\right) + h\left(\frac{x-y}{2}\right) - 2h\left(\frac{x}{2}\right) - h\left(\frac{y}{2}\right) - h\left(-\frac{y}{2}\right)\right)\\
&\qquad + \left(h\left(-\frac{x+y}{2}\right) + h\left(-\frac{x-y}{2}\right) - 2h\left(-\frac{x}{2}\right) - h\left(-\frac{y}{2}\right) - h\left(\frac{y}{2}\right)\right)
\end{aligned}$$

for all $x, y \in X, x+y, x-y \in X$. Hence

$$\begin{aligned}
&\|\mathcal{T}h(x+y) + \mathcal{T}h(x-y) - 2\mathcal{T}h(x) - \mathcal{T}h(y) - \mathcal{T}h(-y)\|\\
&\quad \le 3\left\|h\left(\frac{x+y}{2}\right) + h\left(\frac{x-y}{2}\right) - 2h\left(\frac{x}{2}\right) - h\left(\frac{y}{2}\right) - h\left(-\frac{y}{2}\right)\right\|\\
&\qquad + \left\|h\left(-\frac{x+y}{2}\right) + h\left(-\frac{x-y}{2}\right) - 2h\left(-\frac{x}{2}\right) - h\left(-\frac{y}{2}\right) - h\left(\frac{y}{2}\right)\right\|\\
&\quad \le 3M\left(\left\|\frac{x}{2}\right\|^p + \left\|\frac{y}{2}\right\|^p\right) + M\left(\left\|-\frac{x}{2}\right\|^p + \left\|-\frac{y}{2}\right\|^p\right)\\
&\quad = \frac{4M}{2^p}(\|x\|^p + \|y\|^p).
\end{aligned}$$

Consequently, proceeding by induction we get that if a function $h\colon X \to Y$ satisfies the inequality (6), then

$$\begin{aligned}
&\|\mathcal{T}^n h(x+y) + \mathcal{T}^n h(x-y) - 2\mathcal{T}^n h(x) - \mathcal{T}^n h(y) - \mathcal{T}^n h(-y)\|\\
&\quad \le M\left(\frac{4}{2^p}\right)^n (\|x\|^p + \|y\|^p)
\end{aligned}$$

for all $n \in \mathbb{N}_0$ and $x, y \in X, x+y, x-y \in X$. On account of the above observation and (4)

$$\begin{aligned}
&\|\mathcal{T}^n f(x+y) + \mathcal{T}^n f(x-y) - 2\mathcal{T}^n f(x) - \mathcal{T}^n f(y) - \mathcal{T}^n f(-y)\|\\
&\quad \le c\left(\frac{4}{2^p}\right)^n (\|x\|^p + \|y\|^p)
\end{aligned}$$

for every $n \in \mathbb{N}_0$ and $x, y \in X$ such that $x+y, x-y \in X$. Letting $n \to \infty$ we get

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \qquad x, y \in X,\ x+y, x-y \in X.$$

(2) The idea of the proof is the same as before so we only give a sketch. Replacing y by x in (4) we obtain

$$\left\|\frac{1}{3}f(2x) - \frac{1}{3}f(-x) - f(x)\right\| \le \frac{2c}{3}\|x\|^p, \qquad x \in X. \tag{7}$$

Let functions $\mathcal{T}\colon Y^X \to Y^X$ and $\varepsilon\colon X \to \mathbb{R}_+$ be define by formulas

$$\mathcal{T}\xi(x) = \frac{1}{3}\xi(2x) - \frac{1}{3}\xi(-x), \qquad x \in X,\ \xi \in Y^X$$

and

$$\varepsilon(x) = \frac{2c}{3}\|x\|^p, \qquad x \in X.$$

The inequality (7) takes now the form

$$\|\mathcal{T}f(x) - f(x)\| \le \varepsilon(x), \qquad x \in X.$$

Obviously $\mathcal{T}$ satisfies the inequality (H2) with $f_1(x) = 2x, f_2(x) = -x, L_1(x) = L_2(x) = \frac{1}{3},\ x \in X$. The operator $\Lambda\colon \mathbb{R}_+^X \to \mathbb{R}_+^X$ is given by

$$\Lambda\eta(x) = \frac{1}{3}\eta(2x) + \frac{1}{3}\eta(-x), \qquad x \in X,\ \eta \in \mathbb{R}_+^X.$$

In particular

$$\Lambda\varepsilon(x) = \frac{1}{3}\varepsilon(2x) + \frac{1}{3}\varepsilon(-x) = \frac{2^p+1}{3}\varepsilon(x), \qquad x \in X.$$

Proceeding by induction, we obtain

$$\Lambda^n\varepsilon(x) = \left(\frac{2^p+1}{3}\right)^n \varepsilon(x), \qquad x \in X,\ n \in \mathbb{N}_0.$$

Since $p < 1$, $\frac{2^p+1}{3} < 1$, so the series $\sum\limits_{n=0}^{\infty} \Lambda^n\varepsilon(x)$ is convergent for every $x \in X$ and

$$\varepsilon^*(x) = \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x) = \frac{3}{2-2^p}\varepsilon(x) = \frac{2c}{2-2^p}\|x\|^p, \qquad x \in X.$$

By Theorem 1, there exists a function $g\colon X \to Y$ such

$$g(x) = \lim_{n\to\infty} \mathcal{T}^n f(x), \qquad x \in X,$$

$$g(x) = \frac{1}{3}g(2x) - \frac{1}{3}g(-x), \qquad x \in X$$

and

$$\|f(x) - g(x)\| \le \frac{2c}{2-2^p}\|x\|^p, \qquad x \in X.$$

A trivial verification shows that

$$\begin{aligned}&\|\mathcal{T}^n f(x+y) + \mathcal{T}^n f(x-y) - 2\mathcal{T}^n f(x) - \mathcal{T}^n f(y) - \mathcal{T}^n f(-y)\| \\ &\quad \le \left(\frac{2^p+1}{3}\right)^n c(\|x\|^p + \|y\|^p),\end{aligned}$$

for every $n \in \mathbb{N}_0$ and $x, y \in X$ satisfying $x+y, x-y \in X$. Hence, letting $n \to \infty$ we obtain

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y), \qquad x, y \in X,\ x+y, x-y \in X.$$

(3) In this case let $f_e\colon X \to Y$ and $f_o\colon X \to Y$ be the even and the odd part of the function f, respectively. That means $f_e(x) = \frac{f(x)+f(-x)}{2}, f_o(x) = \frac{f(x)-f(-x)}{2}$ for $x \in X$ and $f = f_e + f_o$. It is easy to see that $f(0) = f_e(0) = f_o(0) = 0$. It follows that

$$\begin{aligned}
&\|f_e(x+y) + f_e(x-y) - 2f_e(x) - f_e(y) - f_e(-y)\| \\
&\quad = \frac{1}{2}\|f(x+y) + f(-x-y) + f(x-y) + f(-x+y) \\
&\qquad - 2f(x) - 2f(-x) - f(y) - f(-y) - f(-y) - f(y)\| \\
&\quad \le \frac{1}{2}(\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \\
&\qquad + \|f(-x-y) + f(-x+y) - 2f(-x) - f(-y) - f(y)\|) \\
&\quad \le c(\|x\|^p + \|y\|^p)
\end{aligned}$$

and analogously

$$\|f_o(x+y) + f_o(x-y) - 2f_o(x) - f_o(y) - f_o(-y)\| \le c(\|x\|^p + \|y\|^p)$$

for every $x, y \in X$ such that $x+y, x-y \in X$. Hence f_e, f_o satisfy the inequality (4).

Replace y by x in (4). By the evenness of f_e,

$$\|f_e(2x) - 4f_e(x)\| \le 2c\|x\|^p, \qquad x \in X$$

which gives

$$\left\| f_e(x) - \frac{f_e(2x)}{4} \right\| \le \frac{1}{2}c\|x\|^p, \qquad x \in X.$$

Let

$$\begin{aligned}
\mathcal{T}\xi(x) &= \frac{\xi(2x)}{4}, \qquad \xi \in Y^X,\ x \in X, \\
\Lambda\delta(x) &= \frac{\delta(2x)}{4}, \qquad \delta \in \mathbb{R}_+^X,\ x \in X
\end{aligned}$$

and $\varepsilon(x) = \frac{1}{2}c\|x\|^p, x \in X$. By Theorem 1, there exists a function $g_e\colon X \to Y$ such that

$$\begin{aligned}
g_e(x) &= \lim_{n\to\infty} \mathcal{T}^n f_e(x), \qquad x \in X, \\
g_e(x) &= \frac{g_e(2x)}{4}, \qquad x \in X
\end{aligned}$$

and

$$\|f_e(x) - g_e(x)\| \le \frac{2c}{4-2^p}\|x\|^p, \qquad x \in X.$$

Moreover,

$$\begin{aligned}&\|\mathcal{T}^n f_e(x+y)+\mathcal{T}^n f_e(x-y)-2\mathcal{T}^n f_e(x)-\mathcal{T}^n f_e(y)-\mathcal{T}^n f_e(-y)\|\\&\quad\le\left(\frac{2^p}{4}\right)^n c(\|x\|^p+\|y\|^p),\end{aligned}$$

for every $n\in\mathbb{N}_0$ and $x,y\in X$ satisfying $x+y,x-y\in X$. Hence g_e satisfies the Drygas equation.

In the same way, replacing y by x in (4) and using the oddness of f_o, we obtain

$$\|f_o(2x)-2f_o(x)\|\le 2c\|x\|^p,\qquad x\in X$$

which with x replacing by $\frac{x}{2}$ yields

$$\left\|f_o(x)-2f_o\left(\frac{x}{2}\right)\right\|\le\frac{2}{2^p}c\|x\|^p,\qquad x\in X.$$

Define now

$$\begin{aligned}\mathcal{T}\xi(x)&=2\xi\left(\frac{x}{2}\right),\qquad \xi\in Y^X,\ x\in X,\\ \Lambda\delta(x)&=2\delta\left(\frac{x}{2}\right),\qquad \delta\in\mathbb{R}_+^X,\ x\in X\end{aligned}$$

and $\varepsilon(x)=\frac{2}{2^p}c\|x\|^p, x\in X$. By Theorem 1, there exists a function $g_o\colon X\to Y$ such that

$$\begin{aligned}g_o(x)&=\lim_{n\to\infty}\mathcal{T}^n f_o(x),\qquad x\in X,\\ g_o(x)&=2g_o\left(\frac{x}{2}\right),\qquad x\in X\end{aligned}$$

and

$$\|f_o(x)-g_o(x)\|\le\frac{2c}{2^p-2}\|x\|^p,\qquad x\in X.$$

By

$$\begin{aligned}&\|\mathcal{T}^n f_o(x+y)+\mathcal{T}^n f_o(x-y)-2\mathcal{T}^n f_o(x)-\mathcal{T}^n f_o(y)-\mathcal{T}^n f_o(-y)\|\\&\quad\le\left(\frac{2}{2^p}\right)^n c(\|x\|^p+\|y\|^p),\qquad n\in\mathbb{N}_0,\ x,y\in X,\ x+y,x-y\in X,\end{aligned}$$

g_o satisfies the Drygas equation. Thus $g=g_e+g_o$ also satisfies the Drygas equation and

$$\|f(x)-g(x)\|\le\frac{2c}{4-2^p}\|x\|^p+\frac{2c}{2^p-2}\|x\|^p,\qquad x\in X.$$

It remains to prove the uniqueness of the function g. We show the case $p > 2$ in details. Let us assume that functions $g_1, g_2 \colon X \to Y$ fulfill the Drygas equation on X and

$$\|f(x) - g_i(x)\| \leq M_i \|x\|^p, \qquad x \in X$$

for some $M_i \geq 0, i = 1, 2$. Hence $\|g_1(x) - g_2(x)\| \leq (M_1 + M_2)\|x\|^p, x \in X$. Since g_1, g_2 satisfy the Drygas equation,

$$g_i(x) = 3g_i\left(\frac{x}{2}\right) + g_i\left(-\frac{x}{2}\right), \qquad x \in X,\ i = 1, 2.$$

Thus

$$\begin{aligned}\|g_1(x) - g_2(x)\| &\leq 3\left\|g_1\left(\frac{x}{2}\right) - g_2\left(\frac{x}{2}\right)\right\| + \left\|g_1\left(-\frac{x}{2}\right) - g_2\left(-\frac{x}{2}\right)\right\| \\ &\leq \frac{4}{2^p}(M_1 + M_2)\|x\|^p\end{aligned}$$

for all $x \in X$. It is easy to check that,

$$\|g_1(x) - g_2(x)\| \leq \left(\frac{4}{2^p}\right)^n (M_1 + M_2)\|x\|^p, \qquad x \in X, n \in \mathbb{N}_0.$$

Letting n to ∞ we obtain

$$g_1(x) = g_2(x), \qquad x \in X.$$

The proofs of the other cases runs as before. □

The following examples show that the assumptions putting on the set X can not be omitted.

Example 3. Let $p > 2, X = (-\infty, -1] \cup \{0\} \cup [1, \infty)$ and $f(x) = |x|, x \in X$. Then

$$|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)| \leq 2(|x|^p + |y|^p)$$

for all $x, y \in X$ such that $x + y, x - y \in X$. Consider functions $g_a \colon X \to \mathbb{R}$ given by $g_a(x) = ax^2, x \in X$, where a is any real constant. The functions g_a obviously satisfy the Drygas equation and

$$|f(x) - g_a(x)| \leq |x|^p, \qquad x \in X$$

for all $a \in [0, 1]$.

Example 4. Let $p \in (0, 1), X = [-1, 1]$ and $f(x) = x^3, x \in X$. Then for all $x, y \in X$ such that $x + y, x - y \in X$

$$|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)| \leq 6(|x|^p + |y|^p).$$

Every function $g_a \colon X \to \mathbb{R}$ given by $g_a(x) = ax, x \in X$ with $a \in \mathbb{R}$ satisfies the Drygas equation and

$$|f(x) - g_a(x)| \leq |x|^p, \qquad x \in X$$

for all $a \in [0, 1]$.

Example 5. Let $1 < p < 2, X = (-\infty, -1] \cup \{0\} \cup [1, \infty)$ and $f(x) = \frac{1}{2}(|x| + x), x \in X$. Then

$$|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)| \leq |x|^p + |y|^p$$

for all $x, y \in X$ such that $x+y, x-y \in X$. Functions $g_a \colon X \to \mathbb{R}$ given by $g_a(x) = ax, x \in X$ are solutions of the Drygas equation and

$$|f(x) - g_a(x)| \leq |x|^p, \qquad x \in X$$

for all $a \in [0, 1]$.

By the same method, we can also obtain the stability result for $p = 0$, but in order to obtain the best known bound we have to make more technical substitutions. The idea is adapted from [7].

Theorem 6. *Let X be such a subset of an Abelian group that $0, -x, 2x, 3x \in X$ for all $x \in X, Y$ a Banach space and $c \geq 0$. If a function $f \colon X \to Y$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq c \tag{8}$$

for all $x, y \in X$ with $x+y, x-y \in X$, then there exists a function $g \colon X \to Y$ satisfying the Drygas equation on X such that

$$\|f(x) - g(x)\| \leq c, \qquad x \in X.$$

Moreover, g is the unique function satisfying equation (3)*, such that $\|f(x) - g(x)\| \leq M, x \in X$ for some $M \geq 0$.*

Proof. Replace (x, y) in (8) by $(2x, x)$, next by $(x, 2x), (-x, -2x)$ and (x, x) (cf. the proof of Theorem 3.2 in [7]). Then

$$\begin{aligned} \|f(3x) - 2f(2x) - f(-x)\| &\leq c, \\ \|f(3x) + f(-x) - 2f(x) - f(2x) - f(-2x)\| &\leq c, \\ \|-f(-3x) - f(x) + 2f(-x) + f(-2x) + f(2x)\| &\leq c, \\ \|f(2x) + f(0) - 3f(x) - f(-x)\| &\leq c, \end{aligned}$$

for $x \in X$. Which with $\|f(0)\| \leq \frac{c}{2}$ give

$$\|2f(3x) - f(-3x) - 9f(x)\| \leq 6c, \qquad x \in X,$$

whence

$$\left\| f(x) - \frac{2}{9}f(3x) + \frac{1}{9}f(-3x) \right\| \leq \frac{2}{3}c, \qquad x \in X. \tag{9}$$

Let functions $\mathcal{T} \colon Y^X \to Y^X$ and $\varepsilon \colon X \to \mathbb{R}_+$ be defined as follows

$$\mathcal{T}\xi(x) = \frac{2}{9}\xi(3x) - \frac{1}{9}\xi(-3x), \qquad x \in X,\ \xi \in Y^X$$

and

$$\varepsilon(x) = \frac{2}{3}c, \qquad x \in X.$$

The inequality (9) now takes the form

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon(x), \qquad x \in X.$$

As before, using Theorem 1, there exists a function $g\colon X \to Y$ such that

$$g(x) = \lim_{n\to\infty} \mathcal{T}^n(x), \qquad x \in X,$$
$$g(x) = \frac{2}{9}g(3x) - \frac{1}{9}g(-3x), \qquad x \in X$$

and

$$\|f(x) - g(x)\| \leq c, \qquad x \in X.$$

In the same manner as in the proofs of Theorem 2 we show that g satisfies the Drygas equation and g is unique. □

3. Nonstability Results

In this section we show that for $p \in \{1, 2\}$ the Drygas equation is not stable. The idea of the construction of the examples comes from the paper [2].

Example 7. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be defined as

$$\phi(x) = \begin{cases} -\alpha, & x \leq -1, \\ \alpha x, & -1 < x < 1, \\ \alpha, & 1 \leq x, \end{cases}$$

where $\alpha > 0$. The function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}, \qquad x \in \mathbb{R}$$

satisfies

$$|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)| \leq 8\alpha(|x| + |y|), \tag{10}$$

but there exist no pair (g, k) of a function $g\colon \mathbb{R} \to \mathbb{R}$ satisfying the Drygas equation and a constant $k \geq 0$ such that

$$|f(x) - g(x)| \leq k|x|, \qquad x \in \mathbb{R}.$$

Proof. We observe that f is odd and bounded by 2α. Now, we show that (10) holds. For $x = y = 0$ and $x, y \in \mathbb{R}$ such that $|x|+|y| \geq 1$ it is obvious. Consider the case $0 < |x| + |y| < 1$. There exists $N \in \mathbb{N}$ such that

$$\frac{1}{2^N} \leq |x| + |y| < \frac{1}{2^{N-1}}.$$

Then $|2^{N-1}x| < 1, |2^{N-1}y| < 1, |2^{N-1}(x+y)| < 1, |2^{N-1}(x-y)| < 1$. Hence

$$2^n x, 2^n y, 2^n(x+y), 2^n(x-y) \in (-1, 1) \quad \text{for } n = 0, 1, \ldots, N-1.$$

By the definition of f,

$$\begin{aligned}
&|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)| \\
&\quad = \left| \sum_{n=0}^{\infty} \frac{\phi(2^n(x+y))}{2^n} + \sum_{n=0}^{\infty} \frac{\phi(2^n(x-y))}{2^n} - 2\sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n} \right| \\
&\quad \le 4\alpha \sum_{n=N}^{\infty} \frac{1}{2^n} = 8\alpha \frac{1}{2^N} \\
&\quad \le 8\alpha(|x|+|y|).
\end{aligned}$$

Assume that there exist a function $g\colon \mathbb{R} \to \mathbb{R}$ satisfying the Drygas equation and a constant $k \ge 0$ such that

$$|f(x)-g(x)| \le k|x|, \qquad x \in \mathbb{R}.$$

Since g fulfills the Drygas equation, there exist an additive function $h\colon \mathbb{R} \to \mathbb{R}$ and a quadratic function $q\colon \mathbb{R} \to \mathbb{R}$ such that $g(x) = h(x) + q(x), x \in \mathbb{R}$. Whence, as f is bounded by 2α, we have

$$|h(x)+q(x)| \le k|x| + 2\alpha, \qquad x \in \mathbb{R}.$$

In particular,

$$|h(nx)+q(nx)| \le k|nx| + 2\alpha, \qquad x \in \mathbb{R},\ n \in \mathbb{N}.$$

The function q satisfies the quadratic functional equation, which implies

$$|h(x)+nq(x)| \le k|x| + \frac{1}{n}2\alpha, \qquad x \in \mathbb{R},\ n \in \mathbb{N},$$

Hence $q(x) = 0, x \in \mathbb{R}$ and

$$|h(x)| \le k|x| + 2\alpha, \qquad x \in \mathbb{R}.$$

It follows that, the additive function h is bounded in the neighborhood of 0, and consequently $h(x) = ax, x \in \mathbb{R}$ for some constant $a \in \mathbb{R}$. Thus

$$|f(x)-ax| \le k|x|, \qquad x \in \mathbb{R}$$

which gives

$$\left|\frac{f(x)}{x}\right| \le k + |a|, \qquad x \in \mathbb{R} \setminus \{0\}. \tag{11}$$

Let N be such that $N\alpha > k + |a|$ and take an $x \in (0, \frac{1}{2^{N-1}})$. Then $2^n x \in (0,1)$ for $n = 0, 1, \ldots, N-1$ and

$$f(x) = \sum_{n=0}^{N-1} \frac{\phi(2^n x)}{2^n} + \sum_{n=N}^{\infty} \frac{\phi(2^n x)}{2^n} > Nx\alpha$$

so

$$\frac{f(x)}{x} > N\alpha > k + |a|,$$

which is contrary to (11). □

For $p = 2$ we have the same example like in the case of the quadratic equation (see [2]).

Example 8. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be defined as

$$\phi(x) = \begin{cases} \alpha, & x \in (-\infty, -1] \cup [1, +\infty), \\ \alpha x^2, & x \in (-1, 1), \end{cases}$$

where $\alpha > 0$. Put

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{4^n}, \qquad x \in \mathbb{R}.$$

Then f satisfies

$$|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)| \le 32\alpha(|x|^2 + |y|^2)$$

and there do not exist a function $g\colon \mathbb{R} \to \mathbb{R}$ satisfying the Drygas equation and a constant $k \ge 0$ such that

$$|f(x) - g(x)| \le k|x|^2, \qquad x \in \mathbb{R}.$$

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